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Product measure steady states of generalized zero range processes

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Abstract

We establish necessary and sufficient conditions for the existence of factorizable steady states of the generalized zero range process on a periodic or infinite lattice. This process allows transitions from a site i to a site $(i + q)$ involving (a bounded number of) multiple particles with rates depending on the content of the site i, the direction q of movement, and the number of particles moving. We also show the sufficiency of a similar condition for the continuous time mass transport process, where the mass at each site and the amount transferred in each transition are continuous variables; we conjecture that this is also a necessary condition.

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1. Introduction

The classical zero range process (ZRP) is a widely studied lattice model with stochastic time evolution [1]. To define the process consider a cubic box $\Lambda \subset \mathbb{Z}^d$ with periodic boundary conditions, i.e. a d-dimensional torus. At each site i of Λ there is a random integer-valued variable $n_i \in \{0, 1, \ldots\}$, representing the number of particles at site i. The time evolution is specified by a function $\alpha_q(n_i)$ giving the rate at which a particle from a site i containing n_i particles jumps to the site $(i + q)$, where q runs over a set of neighbours E (the most common choice is $E = \{\pm e_1, \pm e_2, \ldots\}$, but our treatment holds for any finite E which spans $\mathbb{Z}^d$). The name zero range indicates the fact that the jump rate from i to $(i + q)$ depends only on the number of particles at i.

It is easy to see, when the system is finite, $\alpha_q(n) + \alpha_{-q}(n) \geqslant \delta > 0$ for all $n > 0$ (we always have $\alpha(0) = 0$) and E spans $\mathbb{Z}^d$, that all configurations with a given total particle number $N \equiv \sum_{i\in\Lambda} n_i$ are mutually accessible, and hence there is a unique stationary measure $\tilde{P}_\Lambda(\underline{n}; N)$ for each N. Normalized superpositions of these measures yield all of the stationary states of this system. Conversely, given a stationary measure for which there is a nonzero probability of N particles being present in the system, one can obtain $\tilde{P}_\Lambda(\underline{n}; N)$ by restricting that measure to configurations with N particles.

The ZRP was first introduced in [2]. It was assumed there and in most subsequent works that the rates $\alpha_q(n)$ are of the form

$$\alpha_q(n) = g_q\alpha(n) \tag{1}$$

with g_q independent of n and $\alpha(n)$ independent of q. In this case, the system has the unique steady state given by [2, 3]

$$\tilde{P}_\Lambda(\underline{n}; N) = C_N\delta\left(\sum_{i\in\Lambda} n_i - N\right)\prod_{i\in\Lambda} p(n_i), \tag{2}$$

where

$$p(n) = \frac{c\lambda^n}{\prod_{k=1}^n \alpha(k)}. \tag{3}$$

C_N and c are normalization constants given by

$$c = \left(\sum_{n=0}^{\infty} \frac{\lambda^n}{\prod_{k=1}^n \alpha(k)}\right)^{-1} \tag{4}$$

$$C_N = \left(\sum_{\underline{n}:\sum n_i = N}\prod_{i\in\Lambda} p(n_i)\right)^{-1}. \tag{5}$$

The unique stationary measure $\tilde{P}_\Lambda(\underline{n}; N)$ is thus a restriction to configurations with N particles of the product measure $\prod_{i\in\Lambda} p(n_i)$ with single-site distribution $p(n)$.

In the limit $\Lambda \to \mathbb{Z}^d$ with $N/|\Lambda| \to \rho$ the only stationary extremal measures, i.e. the only stationary measures with a decay of correlations, are product measures with $p(n)$ given in equation (3) as the distribution of single-site occupation numbers [3]. These states are parameterized by λ, which plays the role of the fugacity in an equilibrium system, with different values of λ corresponding to different expected particle densities ρ, where

$$\rho = \sum_{n=1}^{\infty} np(n) = c\sum_{n=1}^{\infty} \frac{n\lambda^n}{\prod_{k=1}^n \alpha(k)}. \tag{6}$$

Recently, there has been a revival of interest in the ZRP. For certain choices of $\alpha(n)$, for example when $\alpha(n) \sim 1 + b/n$ for large n, the ZRP on $\mathbb{Z}$ exhibits a transition between a phase where all sites almost certainly contain finite numbers of particles to a 'condensed' phase where there is a single site containing an infinite number of particles [4, 5]. This condensation has attracted attention as a representative of an interesting class of phase transitions in one-dimensional non-equilibrium systems, and has also been applied to models of growing networks [1, 6].

Evans, Majumdar and Zia [7] have proposed a generalization of the ZRP, called a mass transport model (MTM). They considered a one-dimensional lattice on which there is a continuous 'mass' $m_i \geqslant 0$ at each site, with a parallel update scheme in which at each time step a random mass $\mu_i, 0 \leqslant \mu_i \leqslant m_i$ moves from each site $i \in \mathbb{Z}$ to the neighbouring site $(i+1)$ with a probability density $\phi(\mu, m_i)$. This process shares many features with the (totally asymmetric) zero range process, in particular the existence of a condensation transition in certain cases [8]. One very significant difference from the ZRP, however, is that the system has a product measure steady state if and only if $\phi(\mu, m)$ satisfies a certain condition. Taking a limit in which the probability of a transition at any given site at any given time step goes to zero (discussed in [7]) gives a stochastic process on continuous time (equivalent to a model with random sequential local updates) which we will call a mass transport process or MTP, in which the rate of transitions from site i to $(i+1)$ is given by $\alpha(\mu, m)$. This process has a product measure steady state if and only if there exist functions g and p, such that

$$\alpha(\mu, m) = g(\mu) p(m-\mu)/p(m). \tag{7}$$

An interesting question, then, is whether similar criteria for the existence of a product measure exist for such processes in higher dimension and with movement in both directions allowed. This question is already relevant for the ZRP. A particular case in $d = 2$, studied in [9, 10], has

$$\alpha_{\pm 1}(n) = \alpha[1-\delta_{n,0}] \qquad \alpha_{\pm 2}(n) = \alpha^{(2)}(n) = \alpha n, \tag{8}$$

i.e. a constant rate (independent of n) per occupied site for moving in the $\pm x$ direction and a rate proportional to n in the $\pm y$ direction. A treatment of this system based on fluctuating hydrodynamics and computer simulations (originally conducted on a similar but not quite equivalent system, but which we have reproduced on this system) suggests that this particular system has correlations between occupation numbers at different sites a distance D apart decaying according to a dipole power law D^{-2}. This behaviour, which is very different from a product measure steady state or its projection (2), is conjectured to be generic for non-equilibrium stationary states of systems with non-equilibrium particle conserving dynamics in $\geqslant 2$.

In the present work, we prove rigorously that equation (1) is a necessary and sufficient condition for the existence of product measure steady states for ZRPs, and as a consequence that the system described by (8) has no product measure steady states. This condition in turn is a special case of a condition on a class of systems which we call generalized zero range processes (GZRP), in which we also allow transitions in which more than one particle moves at a time, although we will assume that the number of particles moving in a single transition is bounded. The rate now depends on the number of particles ν which move in the transition as well as the number of particles n at the site before the transition, and so the rates are given by a function $\alpha_q(\nu, n)$ with some $\nu_{\max}$, such that $\alpha_q(n, \nu) = 0$ whenever $\nu > \nu_{\max}$. The classical ZRPs discussed above are a special case with $\nu_{\max} = 1$ and $\alpha_q(1, n) = \alpha_q(n)$. We prove that a necessary and sufficient condition for the GZRP to have product measure steady states is

$$\alpha_q(\nu, n) = \frac{g_q(\nu) f(n-\nu)}{f(n)} \tag{9}$$

for some non-negative $g_q(\nu)$ and $f(n)$ with $\sum f(n) < \infty$. This has a clear similarity to (7), and when $\nu_{\max} = 1$ (9) reduces to (1).

We will prove equation (9) in the course of finding a weaker result for the continuous time mass transport process generalized to dimension $d \geqslant 1$ and to transitions in all directions. We show that these systems have product-measure steady states when $\alpha_q(\mu, m) = g_q(\mu) p(m-\mu)/p(m)$. This condition is also necessary under certain conditions (generalizing (7) to higher dimension) and we conjecture that this is so in all cases.

2. Factorizability in the mass transport process

Let $P_\Lambda(\underline{m}, t)$ be the time-dependent probability density of finding the system in a particular configuration $\underline{m}$ with mass m_i at site $i \in \Lambda$, $m_i \in (0, \infty)$. As noted above, we are considering periodic boundary conditions; this case is somewhat simpler than those of other boundary conditions. The master equation describing the evolution of $P_\Lambda(\underline{m}, t)$ is

$$\frac{\partial P_\Lambda(\underline{m}, t)}{\partial t} = \sum_{i\in\Lambda}\left(-\sum_{q\in E}\int_0^{m_i} \mathrm{d}\mu\, \alpha_q(\mu, m_i) P(\underline{m}, t) + \sum_q \int_0^{m_i} \mathrm{d}\mu\, \alpha_{-q}(\mu, m_{i+q}+\mu) P(\underline{m}^{i,q,\mu}, t)\right), \tag{10}$$

where

$$m_j^{i,q,\mu} = \begin{cases} m_j, & j \notin \{i, i+q\} \\ m_j - \mu, & j = i \\ m_j + \mu, & j = i+q. \end{cases} \tag{11}$$

A stationary state of the system is a distribution $\tilde{P}_\Lambda(\underline{m})$, such that $\partial P_\Lambda(\underline{m}, t)/\partial t = 0$ whenever $P_\Lambda(\underline{m}, t) = \tilde{P}_\Lambda(\underline{m})$, or equivalently

$$\sum_{i\in\Lambda}\sum_q \int_0^{m_i} \mathrm{d}\mu\, \alpha_q(\mu, m_i)\tilde{P}_\Lambda(\underline{m}) = \sum_{i\in\Lambda}\sum_q \int_0^{m_i} \mathrm{d}\mu\, \alpha_{-q}(\mu, m_{i+q}+\mu)\tilde{P}_\Lambda(\underline{m}^{i,q,\mu}). \tag{12}$$

We wish to find conditions under which there is a $\tilde{P}_\Lambda$ which is factorizable, that is which takes the form of

$$\hat{P}_\Lambda(\underline{m}) = \prod_{i\in\Lambda} p(m_i). \tag{13}$$

Assuming that $\alpha_q(\mu, m) > 0$ for all $m > 0$ and $0 < \mu \leqslant m$ for at least one q of each pair of opposite directions, the system in a finite torus Λ will have a unique steady state corresponding to each value of the total mass $M = \sum m_i$. Any linear combination of such states is also a solution of (12). Given a factorizable steady state, states of definite total mass can be obtained by projecting $\hat{P}$ onto the set of configurations with a particular value of M in analogy with equation (2).

Assume that there is a factorizable steady state as in (13). Let $\bar{p}(s)$ be the Laplace transform of $p(m)$, and let

$$\phi_q(\mu, s) = [1/\bar{p}(s)]\int_0^\infty \mathrm{d}m\, \mathrm{e}^{-sm}\alpha_q(\mu, m+\mu)p(m+\mu). \tag{14}$$

Note that, since $\alpha_q(\mu, m) = 0$ for $m < \mu$,

$$\begin{aligned}\int_0^\infty \mathrm{d}m\, \mathrm{e}^{-sm}\alpha_q(\mu, m)p(m) &= \mathrm{e}^{-s\mu}\int_0^\infty \mathrm{d}m\, \mathrm{e}^{-sm}\alpha_q(\mu, m+\mu)p(m+\mu) \\ &= \mathrm{e}^{-s\mu}\phi_q(\mu, s)\bar{p}(s).\end{aligned} \tag{15}$$

We also have

$$\begin{aligned}\int_0^\infty \mathrm{d}m\, \mathrm{e}^{-sm}\int_0^m \mathrm{d}\mu\, \alpha_q(\mu, m)p(m) &= \int_0^\infty \mathrm{d}\mu \int_\mu^\infty \mathrm{d}m\, \mathrm{e}^{-sm}\alpha_q(\mu, m)p(m) \\ &= \int_0^\infty \mathrm{d}\mu\, \mathrm{e}^{-s\mu}\int_0^\infty \mathrm{d}m\, \mathrm{e}^{-sm}\alpha_q(\mu, m+\mu)p(m+\mu) \\ &= \int_0^\infty \mathrm{d}\mu\, \mathrm{e}^{-s\mu}\phi_q(\mu, s)\bar{p}(s).\end{aligned} \tag{16}$$

Multiplying both sides of (12) by $\prod_i \mathrm{e}^{-s_i m_i}$ and integrating over all m_i, we obtain

$$\sum_{i\in\Lambda,q\in E}\left(\prod_{j\neq i}\bar{p}(s_j)\right)\int_0^\infty \mathrm{d}m_i \int_0^{m_i}\mathrm{d}\mu\,\alpha_q(\mu,m_i)p(m_i)\,\mathrm{e}^{-s_i m_i}$$
$$=\sum_{i\in\Lambda,q\in E}\left(\prod_{j\neq i,i+q}\bar{p}(s_j)\right)\int_0^\infty \mathrm{d}m_i\int_0^\infty \mathrm{d}m_{i+q}\int_0^\infty \mathrm{d}\mu$$
$$\times\,\alpha_{-q}(\mu,m_{i+q}+\mu)p(m_i-\mu)p(m_{i+q}+\mu)\exp(-s_i m_i - s_{i+q}m_{i+q}). \qquad (17)$$

Rewriting (17) with the aid of (15) and (16) and cancelling common factors, we obtain

$$\sum_{i\in\Lambda,q}\int_0^\infty \mathrm{d}\mu\,\phi_q(\mu,s_i)\,\mathrm{e}^{-s_i\mu} = \sum_{i\in\Lambda,q}\int_0^\infty \mathrm{d}\mu\,\phi_{-q}(\mu,s_{i+q})\,\mathrm{e}^{-s_i\mu}. \qquad (18)$$

Equation (18) will be satisfied if (though not only if)

$$\phi_q(\mu,s) = g_q(\mu). \qquad (19)$$

In this case equation (14) gives

$$\int_0^\infty \mathrm{d}m\,\mathrm{e}^{-sm}\alpha_q(\mu,m+\mu)p(m+\mu) = g_q(\mu)\int_0^\infty \mathrm{d}m\,\mathrm{e}^{-sm}p(m) \qquad (20)$$

which by uniqueness of the Laplace transform gives

$$\alpha_q(\mu,m) = g_q(\mu)\frac{p(m-\mu)}{p(m)}. \qquad (21)$$

Equation (21) is a generalization of the comparable formula for the unidirectional case [7]. In this case and in all other cases where, for each $q\in E$, either $\alpha_q\equiv 0$ or $\alpha_{-q}\equiv 0$ and hence either $\phi_q\equiv 0$ or $\phi_{-q}\equiv 0$, there is in equation (18) only one term which depends on each pair m_i, m_{i+q}, and in order for the equation to be satisfied it must depend on only one of them. This happens only if (19) holds for that q, so in these cases equation (21) gives the only possible rates for which there is an invariant product measure.

Although in general equation (19) is not the only way of satisfying equation (18), solutions of this equation only correspond to realizable dynamics when p and α_q are non-negative and normalizable; the resulting restrictions on ϕ_q from equation (14) are such that it seems unlikely that there are reasonable (indeed any) rates, other than those in (21), which satisfy all of these conditions.

Dynamics for which the system has a factorizable steady state can be found by beginning with some suitable (positive and normalizable) $p(m)$ and then defining $\alpha_q(\mu,m)$ via (21). For example let

$$p_c(m) = c\,\mathrm{e}^{-cm}\theta(m), \qquad (22)$$

where θ is the Heaviside-step function. The possible transition rates corresponding to $\tilde{P}(\underline{m}) = \prod p_c(m)$ are

$$\alpha_q(m,\mu) = g_q(\mu)\,\mathrm{e}^{c\mu}\theta(m-\mu) = \tilde{g}_q(\mu)\theta(m-\mu), \qquad (23)$$

where $\tilde{g}_q$ are arbitrary non-negative integrable functions, i.e. the rates $\alpha_q(\mu,m)$ are independent of m as long as $\mu\leqslant m$.

3. Reverse processes

In this section we will show that equation (21) is a necessary condition for the existence of factorizable steady states of any MTP whose reverse process is also an MTP. The relevant way in which the reverse process can fail to be an MTP is that it can have transition rates which depend on the mass at the target site of the transition as well as on the mass at the site it is leaving.

In general, given a Markov process with transition rates $K(\underline{m} \to \underline{m}')$ and stationary distribution $\tilde{P}(\underline{m})$, the reverse process is defined by rates $K^*(\underline{m} \to \underline{m}')$ given by

$$K^*(\underline{m} \to \underline{m}') = \frac{K(\underline{m}' \to \underline{m})\tilde{P}(\underline{m}')}{\tilde{P}(\underline{m})}. \tag{24}$$

This new process is what one obtains by running the original process backwards. Consequently the reverse process has the same stationary distribution as the original process, and when K is translation in variant so is K^*.[3]

For an MTP defined by rates $\alpha_q(m_i, \mu)$, $K(\underline{m} \to \underline{m}')$ is equal to $\alpha_q(m_i, \mu)$ for configurations $\underline{m}$ and $\underline{m}'$ related by moving a mass μ from site i to $(i+q)$, and to 0 otherwise. The reverse process is specified by the rate function $\alpha^*_{i,q}(\mu, \underline{m})$ which is the rate of transitions from a configuration $\underline{m}$ in which a mass μ moves from site i to site $(i+q)$; these are the only transitions in this process.

When $\tilde{P}(\underline{m})$ is a product measure with single-site weights $p(m)$, (24) becomes

$$\alpha^*_{i,q}(\mu, \underline{m}) = \alpha_{-q}(\mu, m_{i+q} + \mu)\frac{p(m_i - \mu)p(m_{i+q} + \mu)}{p(m_i)p(m_{i+q})}. \tag{25}$$

Rewriting, we have

$$\alpha^*_{i,q}(\mu, \underline{m})\frac{p(m_i)}{p(m_i - \mu)} = \alpha_{-q}(\mu, m_{i+q} + \mu)\frac{p(m_{i+q} + \mu)}{p(m_{i+q})}. \tag{26}$$

If α^* defines a mass transport process, then it must be independent of all m_j for $j \neq i$. In this case both sides of (26) are equal to some function which depends only on μ and q. This can only be true if α satisfies equation (21), in which case one finds that

$$\alpha^*_{i,q}(\mu, \underline{m}) = \alpha_{-q}(\mu, m_i). \tag{27}$$

4. Factorizability in generalized zero range processes

With mass at each site restricted to an integer particle number n_i, we can reproduce the analysis in the previous section up to equation (18). Denoting the vector of occupation numbers by $\underline{n}$, and the transition rates by $\alpha_q(\nu, n)$, the stationarity condition corresponding to equation (12) is

$$\sum_{i\in\Lambda}\sum_{q\in E}\sum_{\nu=1}^{n_i}(-\alpha_q(\nu, n_i)\tilde{P}_\Lambda(\underline{n}) + \alpha_{-q}(\nu, n_{i+q} + \nu)\tilde{P}_\Lambda(\underline{n}^{i,q,\nu})) = 0. \tag{28}$$

Suppose $\tilde{P}$ is factorizable,

$$\tilde{P}_\Lambda(\underline{n}) = \prod_{i\in\Lambda} p(n_i), \tag{29}$$

[3] If for some configurations $\tilde{P}(\underline{m}) = 0$, then one defines a new process on a configuration space excluding these configurations (this problem does not arise in the case under consideration).

where $p(n)$ is the probability of having n particles at a given site. Then define the generating function (the discrete Laplace transform)

$$\bar{p}(z) = \sum_{n=0}^{\infty} z^n p(n) \tag{30}$$

and let

$$\phi_q(\nu, z) = \frac{\sum_{n=0}^{\infty} z^n \alpha_q(\nu, n+\nu) p(n+\nu)}{\bar{p}(z)}. \tag{31}$$

Note that $\phi_q(\nu, z) \geqslant 0$ for all $\nu, z \geqslant 0$. The counterpart of equation (18) is then

$$\sum_{i\in\Lambda, q\in E} \sum_{\nu=1}^{\infty} z_i^{\nu} \phi_q(\nu, z_i) = \sum_{i\in\Lambda, q\in E} \sum_{\nu=1}^{\infty} z_i^{\nu} \phi_{-q}(\nu, z_{i+q}). \tag{32}$$

We now exploit the assumption that transitions occur only for $\nu \leqslant \nu_{\max}$. Then choosing some $j \in \Lambda$ and $\tilde{q} \in E$ and taking the kth derivative of the above expression with respect to z_j and $z_{j+\tilde{q}}$ gives

$$\sum_{\nu=k}^{\nu_{\max}} \frac{\nu!}{(\nu-k)!} z_j^{\nu-k} \phi_{-\tilde{q}}^{(k)}(\nu, z_{j+\tilde{q}}) + \sum_{\nu=k}^{\nu_{\max}} \frac{\nu!}{(\nu-k)!} z_{j+\tilde{q}}^{\nu-k} \phi_{\tilde{q}}^{(k)}(\nu, z_j) = 0. \tag{33}$$

For $k = \nu_{\max}$, we have

$$\phi_{\tilde{q}}^{(\nu_{\max})}(\nu_{\max}, z_j) + \phi_{-\tilde{q}}^{(\nu_{\max})}(\nu_{\max}, z_{j+\tilde{q}}) = 0. \tag{34}$$

For (34) to hold for all z_j and $z_{j+\tilde{q}}$, both terms on the left-hand side must be constant, and thus the functions $\phi_{\pm\tilde{q}}(\nu_{\max}, \cdot)$ are polynomials of degree $\nu_{\max}$; being non-negative they must have non-negative leading terms. Equation (34) states that pairs of leading terms of these polynomials must add up to zero and so each must be zero, and therefore the functions $\phi_q(\nu_{\max}, \cdot)$ are polynomials of degree at most $(\nu_{\max} - 1)$ for each q.

Now setting $k = \nu_{\max} - 1$, we find by the same reasoning that the functions $\phi_q(\nu_{\max}, \cdot)$ are polynomials of degree at most $\nu_{\max} - 2$. Proceeding in this manner, we find

$$\phi_q(\nu, z) = g_q(\nu) \tag{35}$$

as a necessary as well as a sufficient condition for (32) to be satisfied. Referring to the definition of ϕ, this implies that

$$\alpha_q(\nu, n) = g_q(\nu) \frac{p(n-\nu)}{p(n)} \tag{36}$$

is a necessary and sufficient condition for the existence of a product measure.

In the case where $\nu_{\max} = 1$ and $\alpha_q(1, n) = \alpha_q(n)$, this condition becomes

$$\alpha_q(n) = c_q \frac{p(n-1)}{p(n)}. \tag{37}$$

This is what we referred to above as the classical ZRP, with the well-known stationary measure [2, 4] discussed in the introduction.

5. GZRPs on infinite lattices

In order to show that we have really found all of the factorizable steady states of this class of systems, it remains to be established that the conditions obtained also apply to an infinite lattice; that is, that there are no rates for which the resulting GZRP on an infinite lattice has product measure steady states while the GZRPs defined on finite lattices have none, or do not have the same such stationary states.

Let $P(\underline{n})$ be a product measure with single-site distribution $p(n)$ which is stationary for rate functions α on $\mathbb{Z}^d$, and let Λ be a finite box in $\mathbb{Z}^d$ such that there is some $i_0 \in \Lambda$ such that $i_0 + q \in \Lambda$ for all $q \in E$. Denote by $\underline{n}_\Lambda$ the configuration of the system inside of Λ, and let $P(\underline{n}_\Lambda)$ be the marginal distribution of this configuration. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\underline{n}_\Lambda) = -\sum_{i\in\Lambda}\sum_{q\in E}\sum_{\nu=1}^{n_i} \alpha_q(\nu, n_i) P(\underline{n}_\Lambda) + \sum_{i\in\Lambda}\sum_{q\in E}\sum_{\nu=1}^{n_{i+q}} \alpha_q(\nu, n_i+\nu) P\big(\underline{n}_\Lambda^{i,q,\nu}\big)$$
$$- \sum_{i\in\partial\Lambda}\sum_{q\in E: i+q\in\Lambda}\sum_{\nu=1}^{\infty}\sum_{n=\nu}^{\infty} \alpha_q(\nu, n) P(\underline{n}_\Lambda) p(n)$$
$$+ \sum_{i\in\partial\Lambda}\sum_{q\in E: i+q\in\Lambda}\sum_{n=1}^{\infty}\sum_{\nu=1}^{n} \alpha_q(\nu, n) P\big(\underline{n}_\Lambda^{i,q,\nu}\big) p(n) = 0, \tag{38}$$

where $\partial\Lambda = \{i \in \mathbb{Z}^d \setminus \Lambda | (\exists q \in E)(i+q \in \Lambda)\}$ and

$$n_k^{i,q,\nu} = \begin{cases} n_k, & k \notin \{i, i+q\} \cap \Lambda \\ n_k + \nu, & k = i \in \Lambda \\ n_k - \nu, & k = i+q \in \Lambda. \end{cases} \tag{39}$$

Equation (38) is very similar to equation (28), and by repeating the procedure used above with equation (38) in place of equation (28), it can easily be seen that α and p must satisfy equation (33) and so that (36) is a necessary condition for the existence of a product measure steady state of the process on $\mathbb{Z}^d$ as well as on a finite torus.

6. Conclusion

We have shown that there is a straightforward necessary and sufficient condition, equation (9), for a generalized zero range process to have a product-measure steady state. For mass transport processes, we have found a condition, equation (18), for the existence of a product-measure steady state, which is considerably more opaque than in the GZRP; it is not clear that this is equivalent to the sufficient condition expressed in equation (21), the counterpart of the condition we have obtained for GZRPs. We have, however, presented some reasons to believe that it is.

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